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Constraint Embedding in Kinematics and Dynamics of Hybrid Manipulator Systems

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An approach to kinematics and dynamics based on kinematic influence coefficient matrices is proposed for applications to the analysis and control of motion controlled multibody systems, e.g., hybrid robotic manipulator systems. The scheme is unique in a sense that all the kinematic constraints are completely embedded into the formulations at the kinematics level and equation of motion of the system is obtained in a closed form with respect to the minimal set of independent joint coordinates. Furthermore, all kinematic and dynamic formulations are expressed compactly in the same format by using two special algebraic operators. This isomorphic formalism allows systematic transformations of kinematic and dynamic informations between different sets of coordinates in a purely algebraic way.

Key Words: Hybrid Robotic Manipulator, Multibody Dynamic System, Constraint Embedding, Isomorphism, Transfer of Kinematics and Dynamics

1. Introduction

During last two decades, and important subclass of kinematic and dynamic systems which deal with the behavior of connected rigid bodies undergoing kinemetically planned motions received considerable attention in robotic engineering community as it may be reflected on enormous amount of literatures published in related fields. While most of these efforts were concentrated on robotic manipulators of serial type, hybrid robotic manipulator systems which contain closed kinematic chains in system topology are being proposed to realized systems with larger structural stiffness, required studies on kinematics and dynamics of hybrid robotic manipulator systems become very active recently. In Luh et al.(1985) a simple extension of recursive Newton-Euler scheme was made by using conventional Lagrange's multiplier technique to take kinematic constraints into account. Later in Nakamura(1989) the explicit use of the Lagrange's multiplier was eliminated by directly using the first order constrained relationship among time rate of changes of joint coordinates. Similar approaches can be observed in Rosenberg(1977), Murray et al.(1989), and Wittenberg(1989). An additional care was taken in Murray et al.(1989) to compete with the situation in which sensors for the measurements of joint position and velocity are located at different joints from those actuated(i.e. noncollocated system).

Note that all of these approaches address kinematics and dynamics directly in joint space and provide no explicit ways to relate motion informations of independent joints to Cartesian space counterparts. This fact makes it difficult to apply Cartesian space motion control schemes, which was originally developed for serial robotic manipulators (refer to Khatib, 1987; Whitney, 1969; Luh et al., 1980 and Hogan, 1985), to hybrid type systems. Noting that Cartesian space motion control schemes possess advantage of not requiring inverse position analysis, which is burdensome and inevitable step required in joint space motion control techniques, it is desirable to retain this feature of Cartesian motion control scheme in controlling hybrid type robotic manipulator because the structural complexity of hybrid

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robotic manipulators is generally much higher than serial type ones.

Though Cartesian motion control schemes may remove the burden of performing inverse position analysis, they require rather complete kinematic and dynamic informations of the given manipulator systems. Resolved acceleration scheme(Luh et al., 1980), for example, requires expressions of motions of end-effector up to the second order in terms motions of joints where position and velocity sensors are located. As another example, direct control scheme of dynamics in Cartesian space (Khatib, 1987 and Hogan, 1985) requires dynamics in addition to the second order endeffector motion informations to be expressed in terms of motions of independent joint coordinates.

As pointed out previously, for the varous structural desgn and efficient control of a programmable motion generation mechanisms like robitic manipulators it is essential to express kinematic informations of end-effector (or any point of interest in motion of the system) in compact for in terms of time rate of chages of minimum number of joint coordinates (i. e., generalized coordinates). Although this task becomes trivial for serial type manipulator systems there a number of different ways of choosing generalized coordiantes depending on how the given kinematic constraints are resolved. The kinematic and dynamic modeling algorithm presented in this paper is aimed to answer those problems. The main step used here is to completely resolve the kinematic constraints up to the second order so that both velocity and acceleration informations can be expressed in terms of displacements of properly selected joint coordinates(i.e., a subset of Lagrangian coordinate) and time rate of changes of minimum number of independent joint coordinates in closed forms. The systematic procedure of incorporating constraints into kinematics and dynamics may be called the second order constraint embedding.

Actual procedure of constraint embedding will be performed by using the concept of kinematic influence coefficient (KIC), which was first introduced in dynamic system modeling in Benedict et al.(1978) as the modern equivalent of the foundation work given in Wittenbauer(1923), and since then extensively used in Thomas(1982), Cho et al.(1989), Cho(1989). It is worth noting that the first order external kinematic influence coefficient shares conceptual similarty with the linear and angular partial velocity concept used in Kane et al.(1985). However, the second order external kinematical influence coefficient first introduced by Thomas et al.(1982) has no counterpart in Kane et al.(1985) and has been Proved to be essential in the systematic tracking of acceleration information for multibody dynamic system (Thomas et al., 1982) and also in antagonistic stiffness modeling of redundantly actuated system (Cho et al., 1989 and Yi et al., 1989).

Advantages of the approach developed in this paper are as follows : First of all, since the informations on the displacement of dependent joint coordinate can be readily obtained through measurements in on-line applications or by solving given kniematic constraints in off-line simulations, all the kinematic formulations can be done as if no kinematic constraints are involved in the system. Notice that various cartesian space motion control schemes (Khatib, 1987 ; Whitney, 1969 ; Luh et al., 1980 and Hogan, 1985) and motion planning of kinematically redundant manipulator system (Klein et al., 1983, Yoshikawa et al. and Hollerbach et al., 1987) utilize expresssion of end-effector Jacobian matrix and evaluation of its time rate of changes w.r.t. independent joint coordinates. In view of this observation, approaches proposed in Luh et al.(1985), Nakamura(1989), Murray et al.(1989) and Wittenberg(1989) provide no explicit kinematic informations (e.g., expression of the velocity and acceleration of the end-effector w.r.t. the motion of independent joints) of the systems and address dynamics of the system directly in joint space by assuming required informations on joint motion are obtained by proper means for the given desired end-effector trajectiory. Next, the equation of motion of the system may be readily obtained also in closed form in terms of minimal set of joint coordinates by using commonly employed techniques, e.g., Newton-Euler scheme (Luh et al., 1980), Lagrange's equation(Thomas et al., 1982 and Hollerbach, 1980), and Lagrange's form of d' Alembert's principle (Cho, 1989 and Kane et al., 1985), without any extra attentions being paid to the kinematic constraints. Finally, the idea of the second order constraint embedding enables one, at the levels of generalized velocity and acceleration, to transform, in pureley algebraic fashion, the kinematic and dynamic informations expressed in terms of one set of generalized coordinateds into different set of generalized coordinates(or non-integrable coordinates, i.e., pseudo-coordinates). This benefit comes mainly from the fact that kinematics and dynamics are isomorphically expressed exclusively in terms of generalized velocity and acceleration. From modeling and control point of views, it is necessary that both kinematic and dynamic informations modeled by using one set of generalized coordiantes be easily transformed into another coordinate system(or pseudo-coordinates, e.g., operational space coordinate(Khatib, 1987), so that various modeling schemes can be readily unified.

This paper is organized as follows. In section 2, a systematic procedures for finding the relationships between dependent and independent coordinates are given. Using these relationships kinematics and dynamics problems of constrained multibody dynamic system are addressed in section 3 and section 4 successively. In section 5, transformation technique of kinematic and dynamic informations between different coordinate systems is demonstrated through examples, and finally some conclusions are derived in section 6.

2. Kinematic Constraint Resolution

Consider a multibody dynamic system which contains several closed kinematic chains in its total kinematic structure as shwon in Fig. 1. Note in Fig. 1 that all the joints are assumed to be simple (i.e., it allows only one degree of freedom of relative displacement displacement) without loss of generality and numbered by using positive integers. The set of joint indices used to index all the simple joints in the system will be denoted by

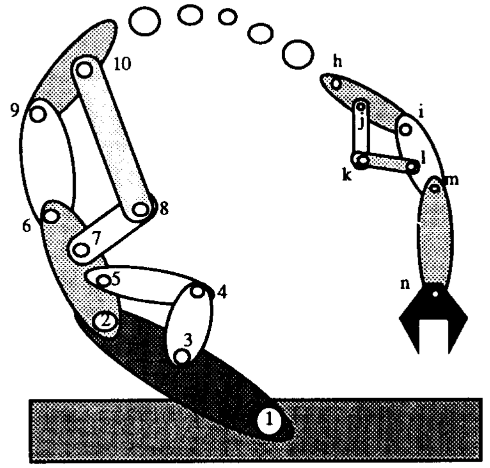

Fig. 1 Topology of a conceptual dynamic system

u, i.e., $u=\{u_i : u_i \in Z^+ \text{ with } i=1, \cdots, J\}$ where Z^+ denotes the set of positive integers and J is the total number of simple joints contained in the system.

A set of independent holonomic constraints in terms of proper Lagrangian coordinates (In this paper, the term Lagrangian coordinates is used to denote any abundant set of coordinates, not necessarily independent) $\boldsymbol{\psi} \in R^J$ may be expressed in vector form as

$$\boldsymbol{f}(\boldsymbol{\psi})=\boldsymbol{0} \tag{1}$$

where $f_i(i=1, \cdots, M)$ are at least twice differentiable functions with respect to (w.r.t.) their arguments $\boldsymbol{\psi}$. Eq. (1) may be rewritten as

$$\boldsymbol{f}(\boldsymbol{\psi}, \boldsymbol{q})=\boldsymbol{0} \tag{2}$$

where $\boldsymbol{\phi} \in R^N$ denotes generalized coordinate vector used to describe the kinematics and dynamics of the system possessing N degree of freedom and vector $\boldsymbol{q} \in R^M$ represents any redundant set of coordinates involved in $\boldsymbol{\psi} \in R^J$ with $J = N+M$. The selections of independent coordinate vector ϕ and dependent coordinated vector $\boldsymbol{q}$ from Lagrangian coordinate vector $\boldsymbol{\psi}$ may be formally described as

$$\phi_i = \psi_{p(i)} \text{ with } i=1, \cdots, N \tag{3}$$

and

$$q_i = \psi_{s(i)} \text{ with } i=1, \cdots, M \tag{4}$$

where two index functions, $p(i)$ and $s(i)$ are defined as

$$p(\cdot):\{1,\cdots,N\}\rightarrow u \tag{5}$$

and

$$s(\cdot):\{1,\cdots,M\}\rightarrow u \tag{6}$$

To facilitate development of algorithm by properly identifying the corresponding components of $\boldsymbol{\phi}$ or $\boldsymbol{q}$ for the given Lagrangian coordinate vector Ψ_i, the inverse relationships $\underline{p}(\cdot)$ and $\underline{s}(\cdot)$ of $p(i)$ and $s(i)$, respectively, are also defined as

$$\underline{p}(\cdot):u\rightarrow\{1,\cdots,N\} \tag{7}$$

$$\underline{s}(\cdot):u\rightarrow\{1,\cdots,C\} \tag{8}$$

The fundamental idea of constraint embedding lies in the simple fact that motions of dependent coordinates, i.e., $\dot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{q}}$ can be found in terms of the counterparts of generalized coordinates $\boldsymbol{\phi}$ by using kinematic constraint equation. This procedure is strainghtforwards and described briefly below.

Differentiation of Eq. (2) w.r.t. time leads

$$\left[\frac{\partial f}{\partial \boldsymbol{\phi}}\right]\dot{\boldsymbol{\phi}}+\left[\frac{\partial f}{\partial \boldsymbol{q}}\right]\dot{\boldsymbol{q}}=\mathbf{0} \tag{9}$$

where $[\partial f/\partial \boldsymbol{\phi}]$ is a matrix of $R^{M\times N}$ whose i^{th} row and j^{th} column element is $\partial f_i/\partial\phi_j$ and $[\partial f/\partial q]$ is a matrix of $R^{M\times M}$ with $\partial f_i/\partial q_j$ as its i^{th} row and j^{th} column element. Solving Eq. (9) for $\dot{\boldsymbol{q}}$, it can be found that

$$\dot{\boldsymbol{q}}=-\left[\frac{\partial f}{\partial \boldsymbol{q}}\right]^{-1}\left[\frac{\partial f}{\partial \boldsymbol{\phi}}\right]\dot{\boldsymbol{\phi}} \tag{10}$$

where the nonsingularity of matrix $[\partial \boldsymbol{f}/\partial \boldsymbol{q}]$ is assumed, and proper selection of generalized coordiantes ϕ and dependent coordiantes $\boldsymbol{q}$ easily ensures this condition due to the independence of constraints.

The first order internal kinematic influence coeffcient (KIC) matrix $[\tilde{G}]\in R^{M\times N}$ of the system is defined in Eq. (10) as

$$[\tilde{\boldsymbol{G}}]\equiv\left[\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\phi}}\right]=-\left[\frac{\partial f}{\partial \boldsymbol{q}}\right]^{-1}\left[\frac{\partial f}{\partial \boldsymbol{\phi}}\right] \tag{12}$$

Using this definition Eq. (10) can be written as

$$\dot{\boldsymbol{q}}=[\tilde{G}]\dot{\boldsymbol{\phi}} \tag{13}$$

The first order internal KIC matrix is usually called Jacobian matrix and its transposed form is frequently used in finding generalized force balance between two coordinates $\boldsymbol{\phi}$ and $\boldsymbol{q}$. Note that although not pusued here, the system with nonlolonomic constraints can be similarly treated by assuming that the first order rate relation between $\boldsymbol{\phi}$ and $\boldsymbol{q}$ may be obtained directly in the form of Eq. (13) without explicit differentiation process.

The second order rate relations between dependent and independent coordinates may be obtained by differentiating Eq. (9) and using Eq. (13) as

$$\ddot{\boldsymbol{g}}=[\tilde{G}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^T\otimes\|\tilde{\boldsymbol{H}}\|\otimes\dot{\boldsymbol{\phi}} \tag{14}$$

where

$$\begin{aligned}\|\tilde{\boldsymbol{H}}\|\equiv\left\|\frac{\partial^2 \boldsymbol{q}}{\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}}\right\|=&-\left[\frac{\partial f}{\partial \boldsymbol{q}}\right]^{-1}\oplus\left\|\frac{\partial^2 \boldsymbol{f}}{\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}}\right\|\\&-[\tilde{G}]^T\otimes\left(\left[\frac{\partial f}{\partial \boldsymbol{q}}\right]^{-1}\oplus\left\|\frac{\partial^2 \boldsymbol{f}}{\partial\boldsymbol{q}\partial\boldsymbol{q}}\right\|\right)\otimes[\tilde{G}]\\&-\left(\left[\frac{\partial f}{\partial \boldsymbol{q}}\right]^{-1}\oplus\left\|\frac{\partial^2 \boldsymbol{f}}{\partial\boldsymbol{\phi}\partial\boldsymbol{q}}\right\|\right)\otimes[\tilde{\boldsymbol{G}}]\\&-[\tilde{\boldsymbol{G}}]^T\otimes\left(\left[\frac{\partial f}{\partial \boldsymbol{q}}\right]^{-1}\oplus\left\|\frac{\partial^2 \boldsymbol{f}}{\partial\boldsymbol{q}\partial\boldsymbol{\phi}}\right\|\right)\end{aligned} \tag{16}$$

In Eqs. (14) and (16), the delimiter $\|\cdot\|$ denotes a three dimensional matrix and three dimensional quadratic operator $\otimes$ and generalized dot product operation $\oplus$ were introduced for the expressional compactness and readiness in computer coding(refer to Appendix for definitions). Notice that the three dimensional matrices introduced in Eq. (16) are constructed such that they are consistent $\oplus$ with $\otimes$ operations, and defined as

$$\left\|\frac{\partial^2 \boldsymbol{f}}{\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}}\right\|_{kij}=\frac{\partial^2 f_k}{\partial\phi_i\partial\phi_j} \tag{17}$$

$$\left\|\frac{\partial^2 \boldsymbol{f}}{\partial\boldsymbol{q}\partial\boldsymbol{\phi}}\right\|_{kij}=\frac{\partial^2 f_k}{\partial q_i\partial\phi_j} \tag{18}$$

$$\left\|\frac{\partial^2 \boldsymbol{f}}{\partial\boldsymbol{q}\partial\boldsymbol{q}}\right\|_{kij}=\frac{\partial^2 f_k}{\partial q_i\partial q_j} \tag{19}$$

and

$$\left\|\frac{\partial^2 \boldsymbol{f}}{\partial\boldsymbol{\phi}\partial\boldsymbol{q}}\right\|_{kij}=\frac{\partial^2 f_k}{\partial\phi_i\partial q_j} \tag{20}$$

where indices k, i, and j denote the corresponding plane, row, and column, respectively.

The three dimensional matrix $\|\tilde{\boldsymbol{H}}\|\in R^{M\times N\times N}$ will be refered to as the second order internal kinematic influence coeffcient matrices of the system. When the set of constraints are holonomic each plane of $\|\tilde{\boldsymbol{H}}\|$ is made of symmetric matrix of $R^{N\times N}$, which explains the fact that it is the generalization of the Hessian matrix defined for a scalar function of a vector variable. When nonholonomic constraints are involved, each plane of $\|\tilde{\boldsymbol{H}}\|$, whose elements are expressed by the first order partial differentiations, is not symmetric in general due to their non-integrable characteristics.

Note also that the definitions on the matrices $[\tilde{G}]$ and $\|\tilde{H}\|$ given respectively in Eq. (12) and (16) involve explicit partial differentiations of the constraints w.r.t. their arguments up to the second order. Actual evaluations of these matrices can be done either by straightforward differentiation of the positional constraint equation according to the definition, which is simple task for the commonly encountered closed kinematic chains formed in a plane, or by starting directly from velocity and acceleration constraint of the closed kinematic chain as in Freeman(1985). When spatially closed kinematic chain is involved in the system topology, the latter approach should by appealing in view of the algebraic complexity of constraint equations constructed through loop-closer method, which may be a discouraging process itself.

3. Spatial Kinematics by Constraint Embedding

Once the first and second internal KIC matrices $[G]$ and $\|\tilde{H}\|$ are found for the system, actual motion of various part of system in three dimensional Cartesian space can be found in terms of motions of independent coordinate ϕ.

Supposing all the joints are properly indexed using positive integers and introducing the concept of virtual cut to virtually break every closed kinematic chain contained in the system, a multibody dynamic system can be considered topologically equivalent to a kinematic tree (Huston et al. 1979). Figure 2 shows a kinematic tree found by applying virtual cuts to the system given in Fig. 1 with some exaggerations made to emphasize cut joints.

In connection with kinematic tree, body indices are assigned as follows. The index 0 is assigned to an inertially fixed body or a body whose motion is prescribed in time. Now, noting that a unique direct path from the body 0 to any point (or a body) can be found by using the kinematic tree, the indices of the remaining bodies can be assigned in such way that they take the same indice as that of the joint which initially meets the body

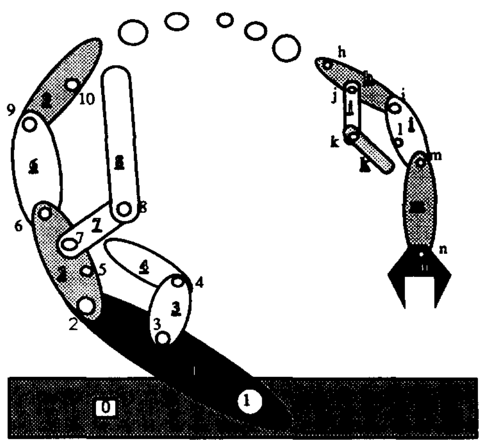

Fig. 2 Spatial Kinematics of constrained system

along the path defined from body 0 to the body of interest. It is important, in this process of assigning body indices, to note that the indices of virtually cut joints play no role in finding body indices, and that the intermediate coordinate frames arising from modeling non-simple joints by a series of simple ones are also assigned proper indices by considering them as imaginary massless bodies.

For easy references two additional sets of indices are defined as follows : set of body indices $v=\{v_i : v_i \in Z \text{ with } i=1, \cdots, K\}$ where Z denotes the set of nonnegative integers and K is the number of bodies involved in kinematic tree, set of link indices $W=\{w_i : w_i \in Z \text{ with } i=1, \cdots, L\}$ where L is the number of physically existing rigid bodies with fixed mass contents. An obvious inclusion relationship among these two sets and the set u introduced in previous section is $W \subseteq V \subseteq U \cap \{0\}$.

The interconnection structure of bodies in a kinematic tree can be fully characterized by a mapping $T(\cdot)$ called connection map (Huston et al.(1979)). The mapping $T(\cdot)$ is designed to operate on the set of body indeces v and describes the connection structure of the bodies in the kinematic tree, such that starting with a particular body index it successively defines the index of the body connected downward along the path.

The above table shows associated connection map $T(\cdot)$ for the kinematic tree described in Fig.

2. The first and second rows totally define the map $T(\cdot)$. For example, it can by easily seen from the table that body 9 is connected to body 0 via bodies 6, 2, and 1.

Basic convention in setting up local coordinate frames is assumed as follows: local z_j-axis is defined fixed on body j, collinear with the axis of rotation or direction of sliding depending on whether the joint j is revolute or prismatic, and $\boldsymbol{x}_j$-and $\boldsymbol{y}_j$-axis fixed in body are properly selected to lie in the plane perpendicular to z_j-axis such that usual dextral rule in satisfied. From now on, all the vectors which appears in equations will be assumed to be expressed w.r.t. the same inertial coordinate frame fixed in body 0.

3.1 Angular velocity

Suppose a body with index k in the system is of concern. Recalling that relative rotational motions between bodies can be collected along appropriate dinematic tree by using associated connection map, the absolute angular velocity of the body k can be expressed in terms of the relative angular velocities between neighboring bodies defined by the connection map as

$$\boldsymbol{\omega}_k=\sum_{j>0}^{k}\boldsymbol{\omega}_j^{T(j)} \tag{21}$$

Note the relative angular velocity of body j w.r.t. body $T(j)$ is given by

$$\boldsymbol{\omega}_j^{T(j)}=\begin{cases}\dot{\psi}_j\boldsymbol{z}_j & \text{if joint } j \text{ is revolute}\\ \boldsymbol{0} & \text{if joint } j \text{ is prismatic}\end{cases} \tag{22}$$

where ψ_j denotes Lagrangian joint coordinate representing proper component of either $\boldsymbol{\phi}$ or $\boldsymbol{q}$. It should be pointed out that the summation in Eq. (21) should be interpreted as being performed downward, starting from body k, along the kinematic tree until the inertially fised body o is reached.

The first order rotational KIC of body k w.r.t. an independent coordinate ϕ_n is defined as a 3×1 vector

$$[G_r^k]_{:n}=\frac{\partial\boldsymbol{\omega}_k}{\partial\phi_n} \tag{23}$$

where the subscript : n is used to denote whole components of the n^{th} column vector of the matrix to which it is attached. Equation (21) implies

$$[G_r^k]_{:n}=\sum_{j>0}^{k}\frac{\partial\dot{\psi}_j}{\partial\dot{\phi}_n}\boldsymbol{z}_j \tag{24}$$

Using Eqs. (11) and (22), three different cases may be distinguished in evaluation $\partial\dot{\psi}_j/\partial\dot{\phi}_n$ as follows:

$$\frac{\partial\dot{\psi}_j}{\partial\dot{\phi}_n}=\begin{cases}0 & \text{if join } j \text{ is prismatic}\\ \delta^j_{p(n)} & \text{if joint } j \text{ is revolute and independent}\\ [\tilde{G}]_{\underline{s}(j)n} & \text{if joint } j \text{ is revolute and dependent}\end{cases} \tag{25}$$

where the Kronecker delta $\delta^j_{p(n)}$ equals 1 if $j=p(n)$ and 0 if $j\neq p(n)$, and $[\tilde{G}]_{\underline{s}(j)n}$ denote $\underline{s}(j)^{th}$ row and n^{th} column element of the first order internal KIC matrix $[\tilde{G}]$, which is equal to $\partial q_{\underline{s}(j)}/\partial\phi_n$ as shown in Eq. (11). Note that when the dynamic system contains no closed kinematic chains, the last case in Eq. (25) is eliminated and the summation in Eq. (24) can be reduced to a single term as in Thomas et al.(1982). Here, without elaborations all possible cases may be expressed in the form of weighted linear combination of vectors $\boldsymbol{z}_j$.

$$\frac{\partial\boldsymbol{\omega}_k}{\partial\phi_n}=\sum_{j>}^{k}\alpha_j\{\beta_j\delta^j_{p(n)}+(1-\beta_j)[\tilde{G}]_{\underline{s}(j)n}\}\boldsymbol{z}_j \tag{26}$$

where α_j equals 1 if joint j is revolute or 0 if joint j is prismatic and β_j equals 1 if joint j is independent, or 0 if joint j is dependent. The final form shown in Eq. (26) gives the expression of the n^{th} column vector of first order external rotational KIC matrices $[G_r^k]\in R^{3\times N}$ associated with body k. Now, the absolute angular velocity of body k can be expressed as

$$\boldsymbol{\omega}_k=\sum_{n=1}^{N}[G_r^k]_{:n}\dot{\phi}_n \tag{27}$$

or in matrix form as

$$\boldsymbol{\omega}_k=[G_r^k]\dot{\boldsymbol{\phi}} \tag{28}$$

3.2 Translational velocity

Suppose a point fixed in body k is of interest and let $\boldsymbol{R}_k$ denote the position vector of the origin of the local coordinate frame fixed in body k w.r.t. a reference coordinate frame fixed in body 0, $\boldsymbol{p}^j$ be the position vector of the point of interest w.r.t. the local coordinate frame fixed in body k, and $[T_k^o]$ be the direction cosine matrix of the local coordinate frame fixed in body k w.r.t. the coordinate frame fixed in body 0. Then, by using kinematic tree position vector $\boldsymbol{p}_k$ of the point of interest w.r.t. reference coordinate frame fixed in body 0 can be expressed as

$$\boldsymbol{P}_k=\boldsymbol{R}_k+[T_k^o]\boldsymbol{p}^k \tag{29}$$

where $\boldsymbol{R}_k$ is assumed to be bulid along the kinematic tree structure. Differentiation this equation in time yields the expression for the absolute linear velocity of the point as

$$\dot{\boldsymbol{P}}_k=\sum_{j>0}^{k}\{\dot{\psi}_j\boldsymbol{z}_j \text{ or } \dot{\psi}_j\boldsymbol{z}_j\times(\boldsymbol{p}_k-\boldsymbol{R}_j)\} \tag{30}$$

where $\boldsymbol{R}_j$ denotes vector measured from origin of inertial coordinate frame fixed in body 0 to orgin of local coordinate frame fixed in body j. Note that since each joint has only one degree of freedom, one of the two terms in Eq. (30) will have effect depending on the joint type. The first order translational KIC of a point fixed in body k w.r.t. a independent coordinate ϕ_n is defined from Eq. (30) as a 3×1 vector

$$[\boldsymbol{G}_t^k]_{:n}=\frac{\partial\dot{\boldsymbol{P}}_k}{\partial\dot{\phi}_n} \tag{31}$$

where

$$\frac{\partial\dot{\boldsymbol{P}}_k}{\partial\dot{\phi}_n}=\sum_{j>0}^{k}\left\{\frac{\partial\dot{\psi}_j}{\partial\dot{\phi}_n}\boldsymbol{z}_j \text{ or } \frac{\partial\dot{\psi}_j}{\partial\dot{\phi}_n}\boldsymbol{z}_j\times(\boldsymbol{p}_k-\boldsymbol{R}_j)\right\} \tag{32}$$

Noting that

$$\frac{\partial\dot{\psi}_j}{\partial\dot{\phi}_n}=\begin{cases}\delta^j_{p(n)} & \text{if joint } j \text{ is independent}\\ [\tilde{G}]_{\underline{s}(j)n} & \text{if joint } j \text{ is dependent}\end{cases} \tag{33}$$

and substituting (33) into Eq. (32) lead to

$$\frac{\partial\dot{\boldsymbol{P}}_k}{\partial\dot{\phi}_n}=\sum_{j>0}^{k}[(1-\alpha_j)\{\beta_j\delta^j_{p(n)}+(1-\beta_j)[\tilde{G}]_{\underline{s}(j)n}\}\boldsymbol{z}_j + \alpha_j)\{\beta_j\delta^j_{p(n)}+(1-\beta_j)[\tilde{G}]_{\underline{s}(j)n}\}\boldsymbol{z}_j\times(\boldsymbol{p}_k-\boldsymbol{R}_j)] \tag{34}$$

where α_j and β_j were defined previously. The final form shown in Eq. (34) gives the expression of the n^{th} column vector of the first order external translational KIC matrix $[\boldsymbol{G}_t^k]\in R^{3\times N}$ associated with a point fixed in body k. Notice that each column of $[\boldsymbol{G}_t^k]$ is formed by weighted summation of vectors $\boldsymbol{z}_j$ and $\boldsymbol{z}_j\times(\boldsymbol{p}_k-\boldsymbol{R}_j)$.

The absolute translational velocity of the point fixed in body k can be expressed in terms of the first order translational KIC matrix as

$$\dot{\boldsymbol{P}}_k=\sum_{n=1}^{N}[\boldsymbol{G}_t^k]_{:n}\dot{\phi}_n \tag{35}$$

or in matrix form as

$$\dot{\boldsymbol{P}}_k=[\boldsymbol{G}_t^k]\dot{\boldsymbol{\phi}} \tag{36}$$

3.3 Angular acceleration

Differentiation of Eq. (27) w.r.t. time produces the expression of the absolute angular acceleration of body k as

$$\boldsymbol{\alpha}_s=\frac{d}{dt}\boldsymbol{\omega}_s \tag{37}$$

$$=\frac{d}{dt}\left(\sum_{n=1}^{N}[\boldsymbol{G}_r^k]_{:n}\dot{\phi}_n\right) \tag{38}$$

$$=\sum_{n=1}^{N}[\boldsymbol{G}_r^k]_{:n}\ddot{\phi}_n+\sum_{n=1}^{N}\frac{d}{dt}([\boldsymbol{G}_r^k]_{:n})\dot{\phi}_n \tag{39}$$

Only the term $d/dt([\boldsymbol{G}_r^k]_{:n})$ involved in the second summation in Eq. (39) requires further treatments as follows: Expand $d/dt([\boldsymbol{G}_r^k]_{:n})$ as

$$\frac{d}{dt}([\boldsymbol{G}_r^k]_{:n})=\sum_{m=1}^{N}\frac{\partial}{\partial\phi_m}([\boldsymbol{G}_r^k]_{:n})\dot{\phi}_m \tag{40}$$

where in view of Eqs. (23) and (26)

$$\frac{\partial}{\partial\phi_m}([\boldsymbol{G}_r^k]_{:n})$$

$$=\frac{\partial}{\partial\phi_m}\left[\sum_{i>0}^{k}\alpha_i\{\beta_i\delta^i_{p(n)}+(1-\beta_i)[\tilde{G}]_{\underline{s}(i)n}\}\boldsymbol{z}_i\right] \tag{41}$$

$$=\sum_{i>0}^{k}\alpha_i\{\beta_i\delta^i_{p(n)}+(1-\beta_i)[\tilde{G}]_{\underline{s}(i)n}\}\frac{\partial\boldsymbol{z}_i}{\partial\phi_m} +\sum_{i>0}^{k}\alpha_i(1-\beta_i)\frac{\partial}{\partial\phi_m}([\tilde{G}]_{\underline{s}(i)n})\boldsymbol{z}_i \tag{42}$$

with

$$\frac{\partial\boldsymbol{z}_i}{\partial\phi_m}=[\boldsymbol{G}_r^i]_{:m}\times\boldsymbol{z}_i \tag{43}$$

$$=[\boldsymbol{G}_r^{T(i)}]_{:m}\times\boldsymbol{z}_i \tag{44}$$

$$=\sum_{j>0}^{T(k)}\alpha_j\{\beta_j\delta^j_m+(1-\beta_i)[\tilde{G}]_{\underline{s}(i)m}\}\boldsymbol{z}_j\times\boldsymbol{z}_i \tag{45}$$

and

$$\frac{\partial}{\partial\phi_m}([\tilde{G}]_{\underline{s}(i)n})=\|\tilde{\boldsymbol{H}}\|_{\underline{s}(i)nm} \tag{46}$$

where $\|\tilde{\boldsymbol{H}}\|_{\underline{s}(i)nm}$ is the $\underline{s}(i)^{th}$ plane, n^{th} row, and m^{th} column element of the matrix $\|\tilde{\boldsymbol{H}}\|$ defined in Eq. (15).

Successive substitutions of Eqs. (40) through (46) into Eq. (39), and some rearrangement, lead us to the final expression of the absolute angular acceleration of the body k as

$$\boldsymbol{\alpha}_k=\sum_{n=1}^{N}[\boldsymbol{G}_r^k]_{:n}\ddot{\phi}_n+\sum_{n=1}^{N}\sum_{m=1}^{N}\|\boldsymbol{H}_r^k\|_{:nm}\dot{\phi}_n\dot{\phi}_m \tag{47}$$

where subscript $:nm$ denotes 3×1 vector made of elements located at n^{th} row and m^{th} column of each plane of a three dimensional matrix. Rewriting Eq. (47) in mtrix form gives

$$\boldsymbol{\alpha}_k=[G_r^k]\ddot{\boldsymbol{\phi}}_n+\dot{\boldsymbol{\phi}}^T\otimes\|\boldsymbol{H}_r^k\|\otimes\dot{\boldsymbol{\phi}} \tag{48}$$

where the scond order rotational KIC matrix $\|\boldsymbol{H}_r^k\|\in R^{3\times N\times N}$ is formed by linear combination of vectors as

$$\begin{aligned}\|\boldsymbol{H}_r^k\|_{:nm}=&\sum_{i>0}^{k}\alpha_i(1-\beta_i)\|\tilde{\boldsymbol{H}}\|_{\underline{s}(i)nm}\boldsymbol{z}_i\\&+\sum_{i>0}^{k}\sum_{j>0}^{k}\alpha_i\alpha_j\{\beta_i\delta_{p(n)}^i+(1-\beta_i)([\tilde{G}]_{\underline{s}(i)n}\}\\&\qquad\{\beta_j\delta_{p(m)}^j+(1-\beta_j)([\tilde{G}]_{\underline{s}(j)m}\}\boldsymbol{z}_j\times\boldsymbol{z}_i\end{aligned} \tag{49}$$

Note that single summation and double summation terms in Eq. (49) respectively shows indirect contributions of motions of dependent joint coordinates q through the second and first order geometric coupling with the motion of independent joint coordinate $\boldsymbol{\phi}$.

3.4 Translational acceleration

To find the absolute translational acceleration of a point fixed in body k in terms of motions of independent joint coordinate $\boldsymbol{\phi}$ costs more efforts than it is required to find angular acceleration. Here, lengthy algebraic details will be omitted and only the result is presented below(refer to [Thomas 1982 and Cho 1989] for complete derivations).

The absolute acceleration of a point fixed in body k can be found as

$$\ddot{\boldsymbol{P}}_k=\sum_{n=1}^{N}[G_t^k]_{:n}\ddot{\phi}_n+\sum_{n=1}^{N}\sum_{m=1}^{N}\|\boldsymbol{H}_t^k\|_{:nm}\dot{\phi}_n\dot{\phi}_m \tag{50}$$

or in matrix form as

$$\ddot{\boldsymbol{P}}_k={}_k[G_t^k]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^T\otimes\|\boldsymbol{H}_t^k\|\otimes\dot{\boldsymbol{\phi}} \tag{51}$$

where the second order external translational KIC matrix $\|\boldsymbol{H}_t^k\|\in R^{3\times N\times N}$ is defined by the collection of vectors as

$$\begin{aligned}\|\boldsymbol{H}_t^k\|_{:nm}=&\frac{\partial}{\partial\phi_m}\left([G_t^k]_{:n}\right)\\=&\sum_{i>0}^{k}\sum_{j>0}^{T(i)}(1-\alpha_i)\alpha_j\{\beta_i\delta_{p(n)}^i+(1-\beta_i)[\tilde{G}]_{\underline{s}(i)n}\}\\&\qquad\{\beta_j\delta_{p(m)}^j+(1-\beta_j)[\tilde{G}]_{\underline{s}(j)m}\}\boldsymbol{z}_j\times\boldsymbol{z}_i\\&+\sum_{i>0}^{k}\sum_{j>0}^{k}\alpha_i(1-\alpha_j)\{\beta_i\delta_{p(n)}^i+(1-\beta_i)[\tilde{G}]_{\underline{s}(i)n}\}\\&\qquad\{\beta_j\delta_{p(m)}^j+(1-\beta_j)[\tilde{G}]_{\underline{s}(j)m}\}\boldsymbol{z}_i\times\boldsymbol{z}_j\\&+\sum_{i>0}^{k}\sum_{j>0}^{T(i)}\alpha_i\alpha_j\{\beta_i\delta_{p(n)}^i+(1-\beta_i)[\tilde{G}]_{\underline{s}(i)n}\}\\&\qquad\{\beta_j\delta_{p(m)}^j+(1-\beta_j)[\tilde{G}]_{\underline{s}(j)m}\}(\boldsymbol{z}_j\times(\boldsymbol{z}_i\times(\boldsymbol{p}_k-\boldsymbol{\phi}_i)))\\&+\sum_{i>0}^{k}\sum_{j>0}^{k}\alpha_i\alpha_j\{\beta_i\delta_{p(n)}^i+(1-\beta_i)[\tilde{G}]_{\underline{s}(i)n}\}\\&\qquad\{\beta_j\delta_{p(m)}^j+(1-\beta_j)[\tilde{G}]_{\underline{s}(j)m}\}(\boldsymbol{z}_i\times(\boldsymbol{z}_j\times(\boldsymbol{p}_k-\boldsymbol{R}_i)))\\&+\sum_{i>0}^{k}\|\tilde{\boldsymbol{H}}\|_{\underline{s}(i)nm}\{(1-\alpha_i)(1-\beta_i)\boldsymbol{z}_i\\&\qquad+\alpha_i(1-\beta_i)\boldsymbol{z}_i\times(\boldsymbol{p}_k-\boldsymbol{R}_i)\}\end{aligned} \tag{52, 53}$$

Notice again that all the first and second order geometric couplings between dependent and independent joint coordinates through kinematic constraint are shown explicitely in the expression.

4. Dynamics

Since all the kinematic information of a constrained dynamic system were compactly expressed in terms of $\dot{\boldsymbol{\phi}}$ and $\ddot{\boldsymbol{\phi}}$ in the previous section, it is direct to find the equation of the motion of the system by using the fundamental equation or Lagrange's form of d'Alembert's principle(Rosenberg, 1977). Here, some additional cares will taken to express the dynamics of the system in an isomorphic form independent of particular coordinate used(see the left-hand side of Eq. (71).

The fundamental equation of body k can be written as

$$\Lambda_k=\int_{V_k}(\ddot{\boldsymbol{P}}dm-fdv)\delta\boldsymbol{P} \tag{54}$$

$$\begin{aligned}&=m_k(\dot{\boldsymbol{P}}^{kc}\delta t)\cdot\ddot{\boldsymbol{P}}^{kc}+(\boldsymbol{\omega}^k\delta t)\cdot(\boldsymbol{I}^k\cdot\boldsymbol{\alpha}^k\\&\quad+\boldsymbol{\omega}_k\times(\boldsymbol{I}^k\cdot\boldsymbol{\omega}_k))\\&\quad-\dot{\boldsymbol{P}}^{kc}\delta t\cdot\boldsymbol{F}^{kt}-\boldsymbol{\omega}^k\delta t\cdot\boldsymbol{T}^{kt}+\delta W_k\end{aligned} \tag{55}$$

where Λ_k represents the total virtual work done on bydy k, $\dot{\boldsymbol{P}}^{kc}$ and $\ddot{\boldsymbol{P}}^{kc}$ denote the absolute velocity and acceleration of the center of mass of body k, respectively, $\boldsymbol{F}^{kt}$ and $\boldsymbol{T}^{kt}$ respectively represent the net force and torque applied externally about the center of the mass, $\boldsymbol{I}^k$ is the inertia dyadics of body k expessed w.r.t. local coordinate frame with its matrix representation being denoted by $[\boldsymbol{I}^k]$, and finally δW_k symbolizes the virtual work done by any internal active forces or/and torque(e.g., actuation force/torque).

Recalling that motion of the bodies in the system can be expressed in terms of independent joint coordinates, the term in Eq. (55) can by rewritten as follows. The first term becomes

$$m_k(\dot{\boldsymbol{P}}^{kc}\delta t)\cdot\ddot{\boldsymbol{P}}^{kc}$$
$$=m_k\delta\boldsymbol{\phi}^T[G_t^{kc}]^T([G_t^{kc}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^T\otimes\|\boldsymbol{H}_t^{kc}\|\otimes\dot{\boldsymbol{\phi}}) \quad (56)$$
$$=m_k\delta\boldsymbol{\phi}^T([G_t^{kc}]^T[G_t^{kc}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^T\otimes\|[G_t^{kc}]^T\oplus\|\boldsymbol{H}_t^{kc}\|\|\otimes\dot{\boldsymbol{\phi}}) \quad (57)$$

The second term can be expressed as

$$(\boldsymbol{\omega}^k\delta t)\cdot(\boldsymbol{I}^k\cdot\boldsymbol{\alpha}^k+\boldsymbol{\omega}^k\times(\boldsymbol{I}^k\cdot\boldsymbol{\omega}^k))$$
$$=\delta\boldsymbol{\phi}^T[G_r^k]^T[\boldsymbol{\Pi}^k]([G_r^k]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^T\otimes\|\boldsymbol{H}_r^k\|\otimes\dot{\boldsymbol{\phi}})+\delta\boldsymbol{\phi}^T[G_r^k]^T(([G_r^k]\dot{\boldsymbol{\phi}})\times([\boldsymbol{\Pi}^k][G_r^k]\dot{\boldsymbol{\phi}})) \quad (58)$$

where the inertially referenced inertial matrix $[\boldsymbol{\Pi}^k]$ for body k is defined by

$$[\boldsymbol{\Pi}^k]=[T_k^0][\boldsymbol{I}^k][T_k^0]^T \quad (59)$$

To arrange this result into a more structured form compatible with $\otimes$ operation, notice that the vector identity $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ implies that

$$[G_r^k]^T(([G_r^k]\dot{\boldsymbol{\phi}})\times([\boldsymbol{\Pi}^k][G_r^k]\dot{\boldsymbol{\phi}}))=([\boldsymbol{\Pi}^k][G_r^k]\dot{\boldsymbol{\phi}})^T([G_r^k]^T\times([G_r^k]\dot{\boldsymbol{\phi}})) \quad (60)$$

where

$$[G_r^k]^T\times([G_r^k]\dot{\boldsymbol{\phi}})=\begin{bmatrix}(\Sigma_i^N[G_r^k]_{:1}\times[G_r^k]_{:i}\dot{\phi}_i)^T\\(\Sigma_i^N[G_r^k]_{:2}\times[G_r^k]_{:i}\dot{\phi}_i)^T\\\vdots\\(\Sigma_i^N[G_r^k]_{:N}\times[G_r^k]_{:i}\dot{\phi}_i)^T\end{bmatrix} \quad (61)$$
$$=\|[G_r^k]\times[G_r^k]\|\otimes\dot{\boldsymbol{\phi}} \quad (62)$$

with the three dimensional matrix $\|[G_r^k]\times[G_r^k]\|$ of $R^{N\times3\times N}$ being defined as

$$\|[G_r^k]\times[G_r^k]\|_{i:j}=[G_r^k]_{:i}\times[G_r^k]_{:j} \quad (63)$$

i.e., j^{th} column vector of i^{th} plane of three dimensional lmatrix $\|[G_r^k]\times[G_r^k]\|$ is formed by cross product of i^{th} and j^{th} column vectors of the first order rotational influence matrix of body k. Using Eq. (60), Eq. (58) can be rewritten as

$$(\boldsymbol{\omega}^k\delta t)\cdot(\boldsymbol{I}^k\cdot\boldsymbol{\alpha}^k+\boldsymbol{\omega}^k\times(\boldsymbol{I}^k\cdot\boldsymbol{\omega}^k))$$
$$=\delta\boldsymbol{\phi}^T\{[G_r^k]^T[\boldsymbol{\Pi}^k][G_r^k]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^T\otimes(([G_r^k]^T[\boldsymbol{\Pi}^k])\oplus\|\boldsymbol{H}_r^k\|+[G_r^k]^T[\boldsymbol{\Pi}^k]\|[G_r^k]\times[G_r^k]\|\otimes\dot{\boldsymbol{\phi}}\} \quad (64)$$

The third and fourth terms in Eq. (55) are expressed as

$$\dot{\boldsymbol{P}}^{kc}\delta t\cdot\boldsymbol{F}^{kt}=\delta\boldsymbol{\phi}^T[G_t^{kc}]^T\boldsymbol{F}^{kt} \quad (65)$$
$$\dot{\boldsymbol{\omega}}^k\delta t\cdot\boldsymbol{T}^{kt}=\delta\boldsymbol{\phi}^T[G_t^k]^T\boldsymbol{T}^{kt} \quad (66)$$

Substituting Eqs. (57) through (66) into Eq. (55) and summing the result over the set of link indices w, we obtain

$$\delta\boldsymbol{\phi}^T\{[\boldsymbol{I}_{\phi\phi}^*]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^T\otimes\|\boldsymbol{P}_{\phi\phi\phi}^*\|\dot{\boldsymbol{\phi}}-\sum_{k\in w}([G_t^{kc}]^T\boldsymbol{F}^{kt}+[G_t^k]^T\boldsymbol{T}^{kt})\}+\sum_{k\in w}\delta W_k=0 \quad (67)$$

where the generalized effective inertia matrix $[\boldsymbol{I}_{\phi\phi}^*]\in R^{N\times N}$ w.r.t. $\boldsymbol{\phi}$ is defined as

$$[\boldsymbol{I}_{\phi\phi}^*]=\sum_{k\in w}(m_k[G_t^{kc}]^T+[G_t^{kc}]+[G_t^k][\boldsymbol{\Pi}^k][G_t^k]) \quad (68)$$

and the matrix $\|\boldsymbol{P}_{\phi\phi\phi}^*\|\in R^{N\times N\times N}$, which explains coriolis and centrifugal forces, is evaluated as

$$\|\boldsymbol{P}_{\phi\phi\phi}^*\|=\sum_{k\in w}\{m_k\|[G_t^{kc}]^T\oplus\|\boldsymbol{H}_t^{kc}\|\,\|+\|([G_t^k]^T[\boldsymbol{\Pi}^k])\oplus\|\boldsymbol{H}_r^k\|\,\|+\|[G_t^k]^T[\boldsymbol{\Pi}^k])\|[G_r^k]\times[G_r^k]\|\,\|\} \quad (69)$$

Finally, noting that net virtual work done by the active internal forces may be expressed as

$$\sum_{k\in w}\delta W_k=\delta\boldsymbol{\phi}^T(\boldsymbol{T}_\phi+[\tilde{G}]^T\boldsymbol{T}q) \quad (70)$$

where $\boldsymbol{T}_\phi$ and $\boldsymbol{T}_q$ are joint torque vectors associated with the primary and secondary joint coordinates of the system, respectively, and also that virtual displacement vector ϕ is independent and arbitrary, we establish the dynamic equation of the system as

$$[\boldsymbol{I}_{\phi\phi}^*]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^T\otimes\|\boldsymbol{P}_{\phi\phi\phi}^*\|\otimes\dot{\boldsymbol{\phi}}=\sum_{k\in w}([G_t^{kc}]^T\boldsymbol{F}^{kt}+[G_r^k]^T\boldsymbol{T}^{kt})+(\boldsymbol{T}_\phi+[\tilde{G}]^T\boldsymbol{T}_q) \quad (71)$$

When $\boldsymbol{T}_q\neq\boldsymbol{0}$, the system is generally under antagonistic operation mode(i.e. superabundance of input forces) and interesting spring-like stiffness properties may be observed due to nonlinear geometric nature of constraint imposed on the system (Cho et al., 1989 and Yi et al., 1990).

5. Isomorphic Transformation in Kinematics and Dynamics

In previous sections, it was shown that all kinematics and dynamics of the system could be expressed in the same structured format in terms of motions of generalized coordinate vector $\boldsymbol{\phi}$.

Here, it is demonstrated by examples that how this isomorphic formalism help one transform informations expressed in one coordinate system into another. Let $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ be two sets of generalized coordinates, and u be any third set of coordinates including pseudo-coordinates whose motions(i.e., $\dot{\boldsymbol{u}}$ and $\ddot{\boldsymbol{u}}$) are to be found.

Consider the case of transformation of kinematic informations. Since it is not possible in general to completely specify system's configurations(i.e., relative positions and orientations of bodies in the system) in terms of a set of nonintegrable pseudo-coordinates, it may be supposed that initial formulations were done in terms of a set of generalized coordinates $\boldsymbol{\phi}$ as

$$\dot{\boldsymbol{u}}=[\boldsymbol{G}_{\phi}^{u}]\dot{\boldsymbol{\phi}}, \tag{72}$$
$$\ddot{\boldsymbol{u}}=[\boldsymbol{G}_{\phi}^{u}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^{T}\otimes\|\boldsymbol{H}_{\phi\phi}^{u}\|\otimes\dot{\boldsymbol{\phi}}, \tag{73}$$

and the first and second order relationships between $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ are known as

$$\dot{\boldsymbol{\theta}}=[\boldsymbol{G}_{\phi}^{\theta}]\dot{\boldsymbol{\phi}}, \tag{74}$$
$$\ddot{\boldsymbol{\theta}}=[\boldsymbol{G}_{\phi}^{\theta}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^{T}\otimes\|\boldsymbol{H}_{\phi\phi}^{\theta}\|\otimes\dot{\boldsymbol{\phi}}, \tag{75}$$

where the Jacobian matrix $[\boldsymbol{G}_{\theta}^{\phi}]$ is assumed to be nonsingular. Now to reestablish motions of $\boldsymbol{u}$ in terms of new set of coordinates $\boldsymbol{\theta}$, Eqs. (74) and (75) are solved for $\dot{\boldsymbol{\theta}}$ and $\ddot{\boldsymbol{\theta}}$ and results are substituted into Eqs. (72) and (73) to yield

$$\dot{\boldsymbol{u}}=[\boldsymbol{G}_{\theta}^{u}]\dot{\boldsymbol{\theta}}, \tag{76}$$
$$\ddot{\boldsymbol{u}}=[\boldsymbol{G}_{\theta}^{u}]\ddot{\boldsymbol{\theta}}+\dot{\boldsymbol{\theta}}^{T}\otimes[\boldsymbol{H}_{\theta\theta}^{u}]\otimes\dot{\boldsymbol{\theta}}, \tag{77}$$

where

$$[\boldsymbol{G}_{\theta}^{u}]=[\boldsymbol{G}_{\phi}^{u}][\boldsymbol{G}_{\phi}^{\theta}]^{-1} \tag{78}$$

and

$$\begin{aligned}\|\boldsymbol{H}_{\theta\theta}^{u}\|=&[\boldsymbol{G}_{\phi}^{\theta}]^{-T}\\ &\otimes\|\,\|\boldsymbol{H}_{\phi\phi}^{u}\|-([\boldsymbol{G}_{\phi}^{u}]\,[\boldsymbol{G}_{\phi}^{\theta}]^{-1})\\ &\oplus\|\boldsymbol{H}_{\phi\phi}^{u}\|\,\|\otimes[\boldsymbol{G}_{\phi}^{\theta}]^{-1}\end{aligned} \tag{79}$$

As examples of transformation of dynamics between different sets of coordinates, two typical cases of particular importance in practical applications are considered. As the first example assume $[\boldsymbol{G}_{\phi}^{u}]$, $[\boldsymbol{H}_{\phi\phi}^{u}]$, and dynamics[1] w.r.t. generalized coordinates $\boldsymbol{\phi}$

$$\boldsymbol{T}_{\phi}=[\boldsymbol{I}_{\phi\phi}^{*}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^{T}\otimes\|\boldsymbol{P}_{\phi\phi\phi}^{*}\|\otimes\dot{\boldsymbol{\phi}} \tag{80}$$

are given and it is desired to find the dynamics w.r.t. $\boldsymbol{u}$. Then, starting from the force balance equation and successively using the first and second order kinematic relations between two generalized coordinates, it is simple to conclude that the dynamics w.r.t. $\boldsymbol{u}$ is given by

$$\boldsymbol{T}_{u}=[\boldsymbol{G}_{\phi}^{u}]^{-T}\boldsymbol{T}_{\phi} \tag{81}$$
$$=[\boldsymbol{G}_{\phi}^{u}]^{-T}([\boldsymbol{I}_{\phi\phi}^{*}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^{T}\otimes\|\boldsymbol{P}_{\phi\phi\phi}^{*}\|\otimes\dot{\boldsymbol{\phi}}) \tag{82}$$
$$=[\boldsymbol{I}_{uu}^{*}]\ddot{\boldsymbol{u}}+\dot{\boldsymbol{u}}^{T}\otimes\|\boldsymbol{P}_{uuu}^{*}\|\otimes\dot{\boldsymbol{u}} \tag{83}$$

where

$$[\boldsymbol{I}_{uu}^{*}]=[\boldsymbol{G}_{\phi}^{u}]^{-T}[\boldsymbol{I}_{\phi\phi}^{*}][\boldsymbol{G}_{\phi}^{u}]^{-1} \tag{84}$$

and

$$\begin{aligned}\|\boldsymbol{P}_{uuu}^{*}\|=&[\boldsymbol{G}_{\phi}^{u}]^{-T}\\ &\otimes\|[\boldsymbol{G}_{\phi}^{u}]^{-T}\oplus\|\boldsymbol{P}_{uuu}^{*}\|-[\boldsymbol{I}_{\phi\phi}^{*}]\\ &\oplus\|\boldsymbol{H}_{\phi\phi}^{u}\|\,\|\otimes[\boldsymbol{G}_{\phi}^{u}]^{-1}\end{aligned} \tag{85}$$

As the second example consider the case of mixed transformation. Given three sets of coordinates ϕ, $\boldsymbol{\theta}$, $\boldsymbol{u}$, $[\boldsymbol{G}_{\phi}^{u}]$, $[\boldsymbol{G}_{\phi}^{\theta}]$, and dynamics w.r.t. ϕ as

$$\boldsymbol{T}_{\phi}=[\boldsymbol{I}_{\phi\phi}^{*}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^{T}\otimes\|\boldsymbol{P}_{\phi\phi\phi}^{*}\|\otimes\dot{\boldsymbol{\phi}} \tag{86}$$

the dynamics w.r.t. mixed coordinates $\boldsymbol{\phi}$, $\boldsymbol{\theta}$, and $\boldsymbol{u}$ may be found follows.

$$\boldsymbol{T}_{\theta}=[\boldsymbol{G}_{\phi}^{\theta}]^{-T}\boldsymbol{T}_{\phi} \tag{87}$$
$$=[\boldsymbol{G}_{\phi}^{\theta}]^{-T}([\boldsymbol{I}_{\phi\phi}^{*}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{\phi}}^{T}\otimes\|\boldsymbol{P}_{\phi\phi\phi}^{*}\|\otimes\dot{\boldsymbol{\phi}}) \tag{88}$$
$$\begin{aligned}=&[\boldsymbol{G}_{\phi}^{\theta}]^{-T}[\boldsymbol{I}_{\phi\phi}^{*}]\ddot{\theta}+([\boldsymbol{G}_{\phi}^{u}]^{-1}\dot{\boldsymbol{u}})^{T}\\ &\otimes\|\boldsymbol{P}_{\phi\phi\phi}^{*}\|\otimes([\boldsymbol{G}_{\phi}^{u}]^{-1}\dot{\boldsymbol{u}})\end{aligned} \tag{89}$$
$$=[\boldsymbol{I}_{\theta\phi}^{*}]\ddot{\boldsymbol{\phi}}+\dot{\boldsymbol{u}}^{T}\otimes[P_{\theta uu}^{*}]\otimes\dot{\boldsymbol{u}} \tag{90}$$

where

$$[\boldsymbol{I}_{\theta\phi}^{*}]=[\boldsymbol{G}_{\phi}^{\theta}]^{-T}([\boldsymbol{I}_{\phi\phi}^{*}] \tag{91}$$

and

$$[\boldsymbol{P}_{\theta uu}^{*}]=[\boldsymbol{G}_{\phi}^{\theta}]^{-T}\otimes\|[\boldsymbol{G}_{\phi}^{\theta}]^{-T}\oplus\|\boldsymbol{P}_{\phi\phi\phi}^{*}\|\,\|\otimes[\boldsymbol{G}_{\phi}^{u}]^{-1} \tag{92}$$

This result may possess particular value in studying the dynamic behavior of the system whose velocities are measured in $\boldsymbol{u}$ coordiantes while input generalized forces are applied at $\boldsymbol{\theta}$ coordiantes.

It is important to point out that the transformations of kinematics or dynamics is possible only in velocity and acceleration levels in general. In other words, the configuration dependent terms $[\boldsymbol{G}]$, $\|\boldsymbol{H}\|$, $[\boldsymbol{I}^{*}]$, and $\|\boldsymbol{P}^{*}\|$ remains to be functions of generalized coordinates initially employed. The reason is obvious when pseudo-coordinates are involved in the transformations. But even in the case of transformations between two sets of generaralized coordinates the situation is unchanged because it is not generally feasible to find global nonlinear relationships in closed form bewteen two sets of generalized coordinates but only the local homeomorphism (Apostol, 1974) through nonsingular Jacobian matrix is possible.

6. Conclusion

In this paper a general kinematic and dynamic modeling algorithm using kinematic influence coefficients was developed. The algorithm is unique yet general in the sense that it provides complete closed form kinematic and dynamic equations isomorphically expressed exclusively in terms of motions(i.e., generalized velocity and acceleration) of the independent joint coordinates. The central tool of this formulation is the constraint embedding through systematic use of interal KIC matrices in kinematics and dynamics.

It is also pointed out that although the formulation given here targeted the holonomically constrained systems, systems involving a class of nonholonomic constraints can be similarly treated by strating directly from velocity constraint relations.

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Appendix

The operational definitions of the operators $\oplus$ and $\otimes$ are provided. Given matrice $[A]\in R^{n\times k}$, $\|B\|\in R^{k\times m\times m}$, new matrix $\|C\|\in R^{n\times m\times m}$ is formed as

$$\|C\|_{i::}=[A]_{i:}\oplus\|B\| \tag{93}$$

$$\equiv\sum_{j=1}^{k}[A]_{ij}\|B\|_{j::} \tag{94}$$

Basically, $\oplus$ operation makes each plane of matrix $\|B\|$ uniformly scaled by the low element of $[A]$ and then summed. Note that $\|B\|\oplus[A]_{i:}$ is not defined.

The operation $\otimes$ is the extension of usual matrix multiplication rule to deal with three dimensional matrix. Typical examples are as follows. A three dimensional quadratic operation is defined for a vector $b\in R^{m}$ and the three dimensional lmatrix $\|B\|\in R^{k\times m\times m}$ to yield another vector $a\in R^{k}$ as

$$a=b^{T}\otimes\|B\|\otimes b \tag{95}$$

$$\equiv\begin{pmatrix} b^{T}\|B\|_{1::}b \\ b^{T}\|B\|_{2::}b \\ \vdots \\ b^{T}\|B\|_{k::}b \end{pmatrix} \tag{96}$$

Multiplication of two matrixes $[A]\in R^{n\times m}$ and $\|B\|\in R^{k\times m\times l}$ may yield $\|C\|\in R^{k\times nl}$ as

$$\|C\|_{i::}=[A]\otimes\|B\| \tag{97}$$

$$\equiv[A]\|B\|_{i::} \tag{98}$$

Similarly $\|B\|\otimes[A]$ is defined with $[A]\in R^{m\times l}$

and $\|\boldsymbol{B}\| \in R^{k\times n\times m}$ as

$$\|\boldsymbol{C}\|_{i::} = \|\boldsymbol{B}\| \otimes [\boldsymbol{A}] \tag{99}$$

$$\equiv \|\boldsymbol{B}\|_{i::}[\boldsymbol{A}] \tag{100}$$

where $\|\boldsymbol{C}\|$ becomes $k\times n\times l$ matrix. No $\otimes$ operation is defined on aly pair of three dimensional matries.